Quantum chaos on a critical Fermi surface

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We compute parameters characterizing many-body quantum chaos for a critical Fermi surface without quasiparticle excitations. We examine a theory of $N$ species of fermions at nonzero density coupled to a $U(1)$ gauge field in two spatial dimensions and determine the Lyapunov rate and the butterfly velocity in an extended random-phase approximation. The thermal diffusivity is found to be universally related to these chaos parameters; i.e., the relationship is independent of $N$, the gauge-coupling constant, the Fermi velocity, the Fermi surface curvature, and high-energy details.

quantum chaos | non-Fermi liquids | thermal transport | quantum criticality

States of quantum matter without quasiparticle excitations are expected (1) to have a shortest-possible local thermalization or phase coherence time of order $\hbar/k_B T$ as $T \to 0$, where $T$ is the absolute temperature. Much recent attention has recently focused on the related and more precise notion of a Lyapunov time, $\tau_L$, the time to many-body quantum chaos (2). By analogy with classical chaos, $\tau_L$ is a measure of the time over which the wavefunction of a quantum system is scrambled by an initial perturbation. This scrambling can be measured by considering the magnitude squared of the commutator of two observables a time $t$ apart (2, 3): The growth of the commutator with $t$ is then a measure of how the quantum state at the later time has been perturbed since the initial observation. In chaotic systems, and with a suitable choice of observables, the growth is initially exponential, $\sim \exp(t/\tau_L)$, and this defines $\tau_L$. With some reasonable physical assumptions on states without quasiparticles, it has been established that this time obeys a lower bound (3)

$$\tau_L \geq \frac{\hbar}{2\pi k_B T}$$

(henceforth, we set $\hbar = \epsilon = 1$). The lower bound is saturated in quantum matter states holographically dual to Einstein gravity (4) and in the Sachdev–Ye–Kitaev model of a strange metal (5–7, *). Relativistic theories in a vector large-$N$ limit provide a weakly coupled realization of states without quasiparticles, and in these cases $\tau_L \sim N/T$ (8–10), which is larger than the bound in Eq. 1 but still of order $1/T$. Fermi liquids have quasiparticles, and their $\tau_L \sim 1/T^2$ is parametrically larger than Eq. 1 as $T \to 0$ (11, 12). In general we expect that $\tau_L$ is of order $1/T$ only in systems without quasiparticle excitations.

In this paper, we turn our attention to non-Fermi liquid states of widespread interest in condensed matter physics. The canonical example we examine is that of $N$ species of fermions at a nonzero density coupled to a $U(1)$ gauge field in two spatial dimensions. Such a theory has a Fermi surface in momentum space that survives in the presence of the gauge field, even though the fermionic quasiparticles do not. (The Fermi surface is defined by the locus of points where the inverse fermion Green’s function vanishes and is typically computed in the gauge $\vec{A} \cdot \vec{a} = 0$: This yields the same Fermi surface as in the closely related problem of a Fermi surface coupled to Ising-nematic order.) Closely related theories apply to a wide class of problems with a critical Fermi surface, and we expect that our results can be extended to these cases too.

It has been recognized for some time (13) that the naive vector $1/N$ expansion of the critical Fermi surface problem breaks down at higher loop orders (beyond three loops in the fermion self-energy). This is in strong contrast to the behavior of relativistic theories at zero density in which the vector $1/N$ expansion is well behaved. This indicates the large $N$ theory of a critical Fermi surface is strongly coupled. Strong-coupling effects have been examined by carefully studying higher loops or by alternative expansion methods (14–17), and in the end the results are similar to those in a random-phase approximation (RPA) theory (18–20). So far, the main additional effect discovered at strong coupling is a small fermion anomalous dimension, but this is not important for our purposes here.

Here, we use an extended RPA theory to compute the Lyapunov time, and the associated butterfly velocity $v_B$ (4, 21–27), for the critical Fermi surface in two spatial dimensions. As $T \to 0$, we find for the Lyapunov rate $\lambda_L \equiv 1/\tau_L$

$$\lambda_L \approx 2.48 T,$$

which obeys the bound $\lambda_L \leq 2\pi T$ in Eq. 1. Notably the value of $\lambda_L$ for the critical Fermi surface is independent of the gauge coupling constant, $\epsilon$, and also of $N$. This supports the conclusion (13) that this theory is strongly coupled in the $N \to \infty$ limit. Our result for the butterfly velocity is more complicated; as $T \to 0$

$$v_B \approx 4.10 \frac{NT^{1/3}}{\exp^{3/3}} \frac{v_F^{1/3}}{\gamma^{1/3}}.$$ 

This depends on both $N$ and $\epsilon$ and also on the Fermi velocity, $v_F$, and the Fermi surface curvature, $\gamma$.

Blake (23, 24) recently suggested, using holographic examples, that there is a universal relation between transport properties, characterized by the energy and charge diffusivities (28) and the parameters characterizing quantum chaos $v_B$ and $\lambda_L$.

\section*{Significance}

All high-temperature superconductors exhibit a remarkable "strange metal" state above their critical temperatures. A theory of the strange metal is a prerequisite for a deeper understanding of high-temperature superconductivity, but the ubiquitous quasiparticle theory of normal metals cannot be extended to the strange metal. Instead, strange metals exhibit many-body chaos over the shortest possible time allowed by quantum theory. We characterize the quantum chaos in a model of fermions at nonzero density coupled to an emergent gauge field. We find a universal relationship between the chaos parameters and the experimentally measurable thermal diffusivity. Our results establish a connection between quantum dephasing and energy transport in states of quantum matter without quasiparticles.
For the critical Fermi surface being studied here, momentum is conserved by the critical theory, and so the electric conductivity is sensitive to additional perturbations that relax momentum (27, 29). However, the thermal conductivity is well defined and finite in the nonrelativistic critical theory (30, 31) even with momentum exactly conserved. So we may define energy diffusivity, $D^E$, which we compute by building upon existing work (18, 32), and find

$$D^E \approx 0.42 \frac{v_F^2}{\lambda_L}. \quad [4]$$

Notably, the factors of $e$, $N$, $v_F$, and $\gamma$ in Eq. 3 cancel precisely in the relationship Eq. 4. This result supports the universality of the relationship between thermal transport and quantum chaos in strongly coupled states without quasiparticles.

A simple intuitive picture of this connection between chaos and transport follows from recognizing that quantum chaos is intimately linked to the loss of phase coherence from electron-electron interactions. As the time derivative of the local phase is determined by the local energy, phase fluctuations and chaos are linked to interaction-induced energy fluctuations and hence to thermal transport. On the other hand, other physical ingredients enter into the transport of other conserved charges, and so we see no reason for a universal connection between chaos and charge transport.

Model

We consider a single patch of a Fermi surface with $N$ fermion flavors, $\psi_j$, coupled to a $U(1)$ gauge boson in two spatial dimensions: This is described by the “chiral non-Fermi liquid” model (33) (Fig. I4). The (Euclidean) action is given by

$$S_\epsilon = \int \frac{d^3k}{(2\pi)^3} \left( \sum_{j=1}^N \psi_j^\dagger(k)(-i\epsilon_0k_0 + \epsilon_k)\psi_j(k) + \frac{N}{2} \phi(k)(c_0|k_0|/|k_0| + k_y^2\phi(-k)) + \epsilon_0 \epsilon_k = v_F k_0 + \gamma k_0^2, \quad c_0 = e^2/(8\pi v_F \gamma). \quad [5]$$

This is derived from the action of a Fermi surface coupled to a $U(1)$ gauge field with gauge coupling constant $e$. We include only the transverse gauge fluctuations in the gauge $\vec{\nabla} \cdot \vec{a} = 0$, in which case the gauge field reduces to a single boson $\phi$ representing the component of the gauge field perpendicular to the Fermi surface. We have already included the one-loop boson self-energy (Fig. S14) in $S_\epsilon$. Unless explicitly mentioned, we set the Fermi velocity $v_F$ and the Fermi surface curvature $\gamma$ to unity in the rest of this work. These factors can be restored by appropriately tracing them through the computations. An advantage of this model is that the one-loop scaling structure of the boson and fermion Green’s functions is “exact.” As there is only a single patch, the one-loop scaling structure is not destroyed by the coupling of different patches at higher loop orders (14). However, this theory is still not fully controllable via the large-$N$ expansion, and IR divergences in higher loop diagrams, such as the three-loop fermion self-energy, enhance their coefficients by powers of $N$. Ultimately, all planar diagrams must be taken into account (13).

A version of this model that combines two antipodal patches of the Fermi surface is amenable to a more controlled $\epsilon = 5/2 - d$ expansion (16). However, our analysis cannot be performed easily with this dimensionally regularized construction, so we restrict ourselves to the $d = 2$ RPA theory. Despite its flaws, the RPA theory has correctly determined other physical features of this theory, such as the scaling of the optical conductivity (20, 31), which agrees with the $\epsilon = 5/2 - d$ expansion (31).

The bare frequency-dependent term in the fermion propagator is irrelevant in the scaling limit and is hence multiplied by the positive infinitesimal $\eta$. However, the presence of this term might lead to crossovers in the quantities that we compute at high temperatures. The above action is invariant under the rescaling

$$k_x \to b^{-1}k_x, \quad k_y \to b^{-1/2}k_y, \quad k_0 \to b^{-3/2}k_0,$$

$$e \to e, \quad \psi \to b^{5/4}\psi, \quad \phi \to b^2\phi. \quad [6]$$

The coupling $e$ is thus dimensionless, and the dynamical critical exponent is $z = 3/2$.

Because we need to perform all computations at finite temperature, it is imperative that we understand what the finite-temperature Green’s functions are. In the above patch theory, the gauge boson does not acquire a thermal mass due to gauge invariance (34). However, we nevertheless add a very small “mass” by hand to use as a regulator. The boson Green’s function then is

$$D(k) = \frac{|k_0|/N}{|k_0|^2 + c_0|k_0| + m^2}. \quad [7]$$

This boson Green’s function may then be used to derive the thermally corrected fermion Green’s function via the one-loop self-energy starting from free fermions (35) (Fig. S1B).

$$G^{-1}(k) = k_0^2 - i\frac{c_f}{N} \text{sgn}(k_0) \frac{T^{2/3}}{H_1/3} \left| \frac{|k_0| - \pi T \text{sgn}(k_0)}{2nT} \right| - \text{sgn}(k_0) \frac{\mu(T)}{N}, \quad \mu(T) = \frac{e^2 T}{3\sqrt{3m^{3/2}}}, \quad c_f = \frac{2^{2/3}}{e^{4/3}}(3\sqrt{3}), \quad [8]$$

where $\mu(T)$ is generated by $m^2$ cutting off an IR divergence coming from the zeroth boson Matsubara frequency, and $H_1/3(x)$ is the analytically continued harmonic number of order $1/3$, with

$$H_n(n \in \mathbb{Z}^+) \equiv \sum_{j=1}^n \frac{1}{j}, \quad H_n(z) = \zeta(r) - \zeta(r, z + 1). \quad [9]$$

This thermally corrected Green’s function is not exact, owing to the uncontrolled nature of the large-$N$ expansion. Higher (three and beyond) loop corrections to the fermion self-energy also contain terms that are ultimately of order $1/N$, which will modify the self-energy but should leave the relative scalings of frequency, momentum, and temperature unchanged (13). The same is also true for various other diagrams. As such, the numerical prefactors in the Lyapunov exponent and butterfly velocity that we determine may not be exact, but we should be able to correctly deduce their scaling properties.
Lyapunov Exponent

To study out-of-time order correlation functions, we define the path integral on a contour \( C \) that runs along both the real and imaginary time directions, with two real-time folds separated by \( i\beta/2 \) (Fig. 1B). The generating functional is given by

\[
Z = \int_C \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}\phi e^{iS[\bar{\psi},\psi,\phi]}.
\]

To measure scrambling, we use fermionic operators, and hence we replace the commutators (2) by anticommutators. We evaluate the index-averaged squared anticommutator (6, 8)

\[
f(t) = \frac{\theta(t)}{\mathcal{N}^2} \sum_{i_j=1}^{N} \int d^2x \langle e^{iHt/2} \{\psi_i(x,t),\psi_j(0)\}\rangle^2 \int d^2x \langle f(t,x) \rangle. \tag{11}
\]

This function is real and invariant under local \( U(1) \) gauge transformations of the \( \psi \)s. The staggered factors of \( e^{-iHt/2} \) place two of the field operators on each of the real-time folds. \( f(t) \) contains the out-of-time ordered correlation function \( \langle \psi_i(x,t)\psi_j(0)\rangle \) that describes scrambling. The anticommutator simplifies the evaluation of correlation function of just the four fermionic operators. \( f(t) \) then measures how the operators “spread” as a function of time. At \( t = 0, \) the anticommutators vanish for \( x \neq 0. \) At later times, the operators become nonlocal under the time evolution, leading to a growth of the function. It is conjectured (3) that at short times

\[
f(t) \sim e^{\lambda_LT} + \ldots, \tag{12}
\]

where \( 0 \leq \lambda_L \leq 2\pi T \) is the Lyapunov exponent. Our goal is to compute \( \lambda_L. \) At long times, which we are not interested in, \( f(t) \) saturates to some finite asymptotic value. Formally, to precisely define \( \lambda_L, \) we need the growing exponential in (12) to have a small prefactor. This can be provided here by examining spatially separated correlators (which we do in the next section), although not by the large-\( N \) limit. Operationally, for now, we compute \( f(t) \) by using diagrams similar to those used in relativistic theories (8).

The approach described in ref. 8 involves summing a series of diagrams to obtain \( f(t). \) The simplest subset of these is a ladder series (Fig. 2), with the “rungs” of the ladder defined on the real-time folds and the “rungs” connecting times separated by \( i\beta/2. \) The interaction vertices are integrated only over the real-time folds as an approximation to minimize technical complexity; more general placements are expected to make corrections to the thermal state that should not affect \( \lambda_L. \) The end result is that one uses retarded Green’s functions for the rails (because the real-time folds involve both forward and backward evolution) and Wightman functions for the rungs (8):

\[
G_R(x,t) = -i\theta(t) \text{Tr} \left[ e^{-\beta H} \{\psi(x,t),\psi^\dagger(0)\}\right]
\]

\[
= -i\theta(t) \text{Tr} \left[ e^{-\beta H} \{\psi(x,t + i\beta/2),\psi^\dagger(0,0,\beta/2)\}\right]
\]

\[
G_R(k) = \frac{1}{k_x + k_y + \frac{\beta}{2\pi} \int \mathcal{N}^2 \frac{T^{2/3} H_{1/3}}{(2\pi)^2} - i \frac{\mu(T_N)}{N}}
\]

\[
D^W(x,t) = \text{Tr} \left[ e^{-\beta H} \{\phi(x,t),\phi(0,0,\beta/2)\}\right]
\]

\[
D^W(k) = \frac{B(k)}{2\sinh^{3/2}(\beta\pi T_N/2)} = \frac{1}{N} \frac{c_0 k_0}{\sinh^{3/2}(\beta\pi T_N/2) + m^2 + c_0^2 k_0^2}.
\]

(B is the boson spectral function). For an explicit derivation of the Wightman functions see SI Appendix B. There are two types of rungs at leading order in \( 1/N: \) one is simply the boson Wightman function. The other is a “box” that contains fermion Wightman functions and retarded boson functions.

The first diagram in the ladder series that has no rungs is given by

\[
f_0(t) = \frac{1}{N} \int d^2x |G_R(x,t)|^2,
\]

\[
f_0(\omega) = \frac{1}{N} \int \frac{d^3k}{(2\pi)^3} G_R^\dagger(k) G_R^*(k - \omega) = \frac{i}{\mathcal{N}^2} \frac{dk_0}{(2\pi)^2} \frac{1}{i \frac{\beta}{2\pi} \int \mathcal{N}^2 \frac{T^{2/3} H_{1/3}}{(2\pi)^2} - i \frac{\mu(T_N)}{N}}
\]

\[
+ 1 + \frac{d^3k'}{(2\pi)^3} \left( -e^2 D^W(k' - k) + K_2(k,k',\omega) \right) f(\omega, k').
\]

This bare term remarkably ends up being \( O(1). \) Because \( m \) is tiny, \( \mu(T) \to +\infty. \) In the time domain, this thus describes a function that decays exponentially very quickly.

We have the Bethe–Salpeter equation of the ladder series

\[
f(\omega) = \frac{1}{\mathcal{N}^2} \int \frac{d^3k}{(2\pi)^3} f(\omega, k) = \frac{1}{\mathcal{N}^2} \int \frac{d^3k}{(2\pi)^3} G_R^\dagger(k) G_R^*(k - \omega)
\]

\[
\times \left[ 1 + \frac{1}{\mathcal{N}^2} \frac{d^3k'}{(2\pi)^3} \left( -e^2 D^W(k' - k) + K_2(k,k',\omega) \right) f(\omega, k') \right].
\]

The \( - \) sign on the \( e^2 \) comes from squaring the factor of \( i \) in the interaction vertex of \( S. \) This factor of \( i \) is not present in the interaction vertex of \( S. \) We need to solve this integral equation to determine the behavior of \( f(t). \) We note that as in ref. 8, the condition for \( f(t) \) to grow exponentially is that the ladder sum be invariant under the addition of an extra unit to the ladder; that is,

\[
f(\omega, k) = G_R^\dagger(k) G_R^*(k - \omega)
\]

\[
\times \int \frac{d^3k'}{(2\pi)^3} \left( -e^2 D^W(k' - k) + K_2(k,k',\omega) \right) f(\omega, k').
\]

We have

\[
K_2(k,k',\omega) = N e^4 \int \frac{d^3k_1}{(2\pi)^3} D_R(k_1) D_R^* (\omega - k_1)
\]

\[
\times G_0^W(k - k_1) G_0^W(k' - k_1).
\]

Here, we use the bare fermion Wightman functions, as the self-energy corrections will come in at higher orders in \( 1/N. \) As the integral is free of IR divergences, the overall power of \( 1/N \) in this contribution is not enhanced and this simplification should be safe. In the bare fermion Wightman functions, we drop the
frequency-dependent term that is irrelevant at low frequencies by sending \( \eta \to 0 \), to preserve the quantum critical scaling

\[
G^W_0(k) = \frac{A(k)}{2 \cosh \frac{2k}{T}} \to \pi \delta(k_x + k_y^2).
\]  

\[ 18 \]

\((A) is the fermion spectral function.\) There is a \( \cosh \) instead of a sinh in the fermion Wightman function (SI Appendix B). We then have

\[
K_2(k, k', \omega) = \frac{e^4}{N} \int \frac{d^2k}{(2\pi)^2} \frac{(k_x - k'_x)^2}{|k_x - k'_x|^3} \frac{1}{|k_y - k'_y|^3} \frac{1}{8 \pi \epsilon_0 k_0 (k_x - k'_x)^2} \cosh \frac{\beta k_x}{2} \cosh \frac{\beta k'_x}{2}.
\]

\[ 19 \]

Because there are no IR divergences, we drop the \( m_2^2 \) s. Doing the \( k_{1x} \) integral followed by the \( k_{1y} \) one, this simplifies to

\[
K_2(k, k', \omega) = \frac{e^4}{N} \int \frac{d^2k}{(2\pi)^2} \frac{| \epsilon_k - \epsilon_{k'} \rangle \langle \epsilon_k - \epsilon_{k'} |}{|k_x - k'_x|^3} \frac{1}{|k_y - k'_y|^3} \frac{1}{8 \pi \epsilon_0 k_0 (k_x - k'_x)^2} \cosh \frac{\beta k_x}{2} \cosh \frac{\beta k'_x}{2}.
\]

\[ 20 \]

Due to the sliding symmetry along the Fermi surface (14), we expect the eigenfunction that we are interested in to obey \( f(\omega, k) = f(\omega, k_0, k_1) \). This can be proved by induction, considering the series of diagrams that we sum. We can then shift \( k'_y \to k_y - k_y^2 \) followed by \( k'_y \to k_y + k_y^2 \) and integrate over \( k'_y \),

\[
\int \frac{d^2k'}{(2\pi)^2} K_2(k, k', \omega) f(\omega, k') = \frac{e^4}{24 \pi \sqrt{3} \epsilon_0^3 N} \int \frac{dk_1^0 d\epsilon_k}{(2\pi)^2} \int \frac{dk_1}{(2\pi)^2} \frac{| \epsilon_k - \epsilon_{k'} \rangle \langle \epsilon_k - \epsilon_{k'} |}{|k_x - k'_x|^3} \frac{1}{|k_y - k'_y|^3} \frac{1}{8 \pi \epsilon_0 k_0 (k_x - k'_x)^2} \cosh \frac{\beta k_x}{2} \cosh \frac{\beta k'_x}{2}.
\]

\[ 21 \]

and

\[
e^{-2} \int \frac{d^2k'}{(2\pi)^2} D^W(k - k') f(\omega, k')
\]

\[
e^{-2} \int \frac{d^2k'}{(2\pi)^2} D^W(k - k') f(\omega, k')
\]

\[
= \frac{e^2}{N} \frac{2\mu(T)}{e^2} + \lim_{m \to 0} \int \frac{d^2k'}{(2\pi)^2} \left( \frac{1}{|k_y - k'_y|^3} \frac{1}{8 \pi \epsilon_0 k_0 (k_x - k'_x)^2} \cosh \frac{\beta k_x}{2} \cosh \frac{\beta k'_x}{2} \right)
\]

\[ 22 \]

where we added and subtracted terms to make the IR divergences explicit. If we expand the numerator of the integrand in the above for \( k'_y \to k_y \), we see that the integral is finite and free of IR divergences.

Interestingly, both pieces of the kernel no longer depend on \( k_x \) and \( k'_x \). Thus, we can integrate both sides of the equation over \( k_x \) and \( k'_x \) to get an equation for \( f(\omega, k_0) \equiv \int \frac{d^2k}{(2\pi)^2} f(\omega, k_0, k_1) \). From Eq. 14, we can see that the IR divergent piece \( \propto \mu(T) \) cancels out. The dependence on \( N \) also cancels out. We finally get

\[
e^{-2} \lim_{m \to 0} \int \frac{dk_1^0 d\epsilon_k}{(2\pi)^2} \frac{| \epsilon_k - \epsilon_{k'} \rangle \langle \epsilon_k - \epsilon_{k'} |}{|k_y - k'_y|^3} \frac{1}{8 \pi \epsilon_0 k_0 (k_x - k'_x)^2} \cosh \frac{\beta k_x}{2} \cosh \frac{\beta k'_x}{2}.
\]

\[ 23 \]

As a matrix equation, this is of the form \( M(\omega) \tilde{f}(\omega) = 0 \). Because we are looking for a positive growth exponent, we need to numerically find solutions of this equation on the positive imaginary \( \omega \) axis. The analytic continuations of the self-energies that we made are still valid as long as \( \text{Im}[\omega] > 0 \). The largest solution will provide the growth exponent \( \lambda_c \). We can see from the above equation and from the quantum critical scaling \( k_0, k'_0 \sim T \), \( k_0, k'_0 \sim e^{2/3} T^{2/3} \) that \( \lambda_c \propto T \) and is independent of \( e \). The numerical solution to this equation is detailed in SI Appendix D. We find that \( \lambda_c \approx 2.48 T \), which is well within the bound of ref. 3. We further see that \( \lambda_c \) is not suppressed by powers of \( 1/N \), unlike other vector models in the large-\( N \) limit. This result indicates that this theory is strongly coupled at the lowest-energy scales, even for large values of \( N \).

At high temperatures, when \( NT^{1/3} / e^{2/3} \gg 1 \), we may no longer be able to neglect the bare frequency-dependent term in the fermion propagators. This would essentially amount to adding a term \( \sim N \omega f(\omega, k_0) \) to the right-hand side of Eq. 23. Counting powers, we then might expect \( \lambda_c \sim e^{2/3} T^{2/3}/N \). In SI Appendix C we consider a few higher order (in \( 1/N \)) corrections to the ladder series (Fig. S2) and show that some of them are insignificant.

**Butterfly Velocity**

The out-of-time order correlation function evaluated at spatially separated points characterizes the divergence of phase space trajectories in both space and time. This process is described by the function \( f(t, x) \) defined in Eq. 11, which is the same as the function \( f(t) \) we used to determine \( \lambda_c \), except for the integration over spatial coordinates. This function should contain a traveling-wave term that propagates with a speed known as the “butterfly velocity” (25). To compute this function we need to evaluate the ladder diagrams at a finite external momentum \( p \). For simplicity, and because the component of the Fermi velocity perpendicular to the Fermi surface dominates the one parallel to the Fermi surface, we take the external momentum to also be perpendicular to the Fermi surface. This allows us to determine the component of the butterfly velocity perpendicular to the Fermi surface (\( v_{\perp 1} \)).

Repeating the same steps that led to the derivation of Eq. 23, we simply obtain

\[
e^{-2} \lim_{m \to 0} \int \frac{dk_1^0 d\epsilon_k}{(2\pi)^2} \frac{| \epsilon_k - \epsilon_{k'} \rangle \langle \epsilon_k - \epsilon_{k'} |}{|k_y - k'_y|^3} \frac{1}{8 \pi \epsilon_0 k_0 (k_x - k'_x)^2} \cosh \frac{\beta k_x}{2} \cosh \frac{\beta k'_x}{2}.
\]

\[ 24 \]
For small $p_s$, we expect the change in exponent $\delta \lambda_L / T \sim -iNp_s / (e^{4/3} T^{2/3})$. This implies that

$$f(t, x) \sim e^{\delta \lambda L / T^2} \int \frac{dp_s}{2\pi} g(t, Np_s) e^{ip_s(x-v_{\parallel}t)} \sim \frac{NT^{1/3}}{e^{4/3}}.$$  \hfill [25]

The structure of the above equation indicates that chaos propagates as a wave pulse that travels at the butterfly velocity. The wave pulse is not a soliton and broadens as it moves (25); this is encoded in the function $g(t, Np_s)$ and further details are provided in SI Appendix D. Note that this shows $v_{\parallel} \sim T^{1-1/3}$, which can also be straightforwardly derived by using the appropriate scalings of space and time, $|x| = -1$ and $|t| = -z$, and is also seen in holographic models (23). Numerically we find that $\delta \lambda_L / T \approx -4.10(2iNp_s / (e^{4/3} T^{2/3}))$, giving the result of Eq. 3 once the factors of Fermi velocity $v_F$ and Fermi surface curvature $\gamma$ are restored (SI Appendix D).

This is again strictly valid only at the lowest temperatures, where $NT^{1/3} / e^{4/3} \ll 1$. Thus, the butterfly velocity cannot be arbitrarily large in the large-$N$ limit. When $NT^{1/3} / e^{4/3} \sim 1$, the structure of the fermion propagator indicates that there may be a crossover to a $z = 1$ regime, in which $\varphi_{\parallel} \ll 1$ becomes a constant independent of $N$ and $T$.

With the scalings $[y] = -1/2$ and $[t] = -z$, we see that the component parallel to the Fermi surface, $v_{\perp} \sim T^{2/3}$, which is smaller than $v_{\parallel}$ at low temperatures. Then the butterfly effect will be dominated by propagation perpendicular to the Fermi surface in the scaling limit.

Energy Diffusion
It has been conjectured, and shown in holographic models (23, 28), that the butterfly effect controls diffusive transport. The thermal diffusivity

$$D^E = \frac{\kappa}{C_V} \sim \frac{v_F^2}{2\pi T},$$  \hfill [26]

where $\kappa$ is the thermal conductivity and $C_V$ is the specific heat at fixed density. In holographic theories $\lambda_L = 2\pi T$, so a more appropriate phrasing of the above equation is $D^E \sim v_F^2 / \lambda_L$ (36).

We can compute $C_V$ using the free energy of the fermions [the contribution of the boson is expected to be subleading at low temperatures (31)]

$$F = -NT \sum_{k_0} \int \frac{d^2k}{(2\pi)^2} \ln \tilde{G}^{-1}(k), \quad C_V = -T \frac{\partial^2 F}{\partial T^2},$$  \hfill [27]

where we use the one-loop dressed fermion propagator at zero temperature (14),

$$\tilde{G}^{-1}(k) = k^2 + k_y^2 - i \frac{\psi}{N} \text{sgn}(k_0) |k_0|^{2/3}, \quad \psi = \frac{3e}{2(2\pi)^{2/3}}.$$  \hfill [28]

This computation is carried out in SI Appendix E. We obtain

$$C_V = \frac{10(2^{2/3} - 1)}{9(2\pi)^{2/3}} \Gamma(5/3) \zeta(5/3) T^{2/3} e^{4/3} \gamma_3^{1/3} \frac{v_F^{10/3}}{\lambda_L} \int \frac{dk_0}{2\pi}.$$  \hfill [29]

where we have again restored the factors of $v_F$ and $\gamma$. Because of the theorem of a single Fermi surface patch is chiral, currents are nonzero even in equilibrium. We must thus define conductivities with respect to the additional change in these currents when electric fields and temperature gradients are applied. The thermal conductivity $\kappa$ is finite in the DC limit as it is defined under conditions where no additional electrical current flows. This can be achieved by simultaneously applying an electric field and a temperature gradient such that there is only an additional energy current but no additional electrical current. We have

$$\kappa = \frac{e^2 T}{\sigma},$$  \hfill [30]

where $\kappa$, $\sigma$ are the thermoelectric conductivities and $\sigma$ is the electrical conductivity. The infinities in the DC limit cancel between $\kappa$ and the other term, yielding a finite $\kappa$ (31). $\kappa$ may be obtained from the Kubo formula (37)

$$\kappa_{\perp} = -\beta \text{Re} \left[ \lim_{\omega \to 0} \left( \frac{\partial}{\partial \omega} i(J_\perp^E J_\perp^F) \left( i\theta \rightarrow \omega + i0^+ \right) \right) \right],$$  \hfill [31]

with the energy current

$$J_\perp^E(x_0) = -i \int \frac{d^3k}{(2\pi)^3} \left( k_0 + \frac{\psi}{2} \right) \frac{\partial \psi}{\partial k_0} \psi^\dagger(k + \psi_0) \psi(k).$$  \hfill [32]

We compute the conductivities using the one-loop dressed fermion propagators in SI Appendix E (the boson again does not contribute directly due to the absence of an $x$-dependent term in its dispersion). The simplest vertex correction vanishes due to the structure of the fermion dispersions and other corrections are in general suppressed by powers of $N$. In this approximation $\kappa_{\perp}$ is finite and $\alpha_{\perp} \propto \left( J_\perp^F J_\perp^F \right)$ (where $J_\perp$ is the charge current) vanishes, so $\kappa_{\parallel} = \kappa_{\perp}$. Note that, in reality, $\kappa$, $\sigma$, $\gamma$ would all be infinite and their combination into $\kappa$ would be finite, but the final finite value of $\kappa$ should be qualitatively similar to the value obtained from our approximation. We obtain (restoring $v_F$ and $\gamma$)

$$\kappa_{\parallel} \approx 0.28 \frac{N^2 T^{1/3}}{e^2} \frac{\gamma_3^{1/3}}{\gamma_3^{1/3}} \int \frac{dk_0}{2\pi}.$$  \hfill [33]

Using Eqs. 2, 3, 26, and 29 we then see that

$$D^E \approx 2.83 \frac{N^2}{e^{8/3} T^{1/3}} \frac{v_F^{10/3}}{\lambda_L} \approx 0.42 \frac{v_{\parallel}^2}{\lambda_L}.$$  \hfill [34]

The factors of powers of $T$, $N$, and $e$ match exactly on both sides of the equation and the constant of proportionality between $D^E$ and $v_{\parallel}^2 / \lambda_L$ is an $O(1)$ number. This result strongly indicates that the butterfly effect is responsible for diffusive energy transport in this theory. The DC electrical conductivity is, however, infinite due to translational invariance, and hence, unfortunately, such a statement cannot be made for charge transport in this model. Note that the hyperscaling violating factor $\int \frac{dk_0}{2\pi}$ (31) cancels between $\kappa_{\parallel}$ and $C_V$. However, if we consider $\kappa_{\parallel}$, this does not happen due to the additional $k_0$ dependence in $J_\parallel^E$. Thus, $D^E$ will not be given by $v_{\parallel}^2 / \lambda_L$.

**Outlook**
We have computed the Lyapunov exponent $\lambda_L$ and butterfly velocity $v_{\parallel}$ for a single patch of a Fermi surface with $N$ fermion flavors coupled to a U(1) gauge field. At the lowest-energy scales, this theory is strongly coupled regardless of the value of $N$, and we hence find that $\lambda_L$ is independent of $N$ to leading order in $1/N$. The proposed universal bound of $\lambda_L \leq 2\pi T$ is also obeyed. Although the $1/N$ expansion is not fully controllable, it has nevertheless been capable of correctly determining many physical features of this theory in the past. We find that the butterfly velocity is dominated by propagation perpendicular to the Fermi surface and $v_{\parallel} / \lambda_L \sim NT^{1/3}$. Most interestingly, we find that the butterfly effect controls diffusive transport in this model, with the thermal diffusivity $D^E \propto v_{\parallel}^2 / \lambda_L$. Our results are valid
at the lowest-energy scales, at which the quantum critical scaling holds. At high temperatures, we might expect $\lambda L$ to cross over to a slower $T^{2/3}/N$ scaling and that $\nu_{BL}$ simply becomes a constant independent of $N$ and $T$. Although technically much more complex to obtain, it would be interesting to compare the results derived from a more controlled calculation, such as the $\epsilon = 5/2 - d$ expansion for the two-patch version of the problem, with our results. Finally, we note recent experimental measurements of thermal diffusivity in the cuprates (38), which find a strong coupling to phonons. It would be of interest to extend the chaos theories to include the electron–phonon coupling.

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