GROUP OF ISOMORPHISMS OF AN ABELIAN GROUP

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Communicated April 21, 1930

It is well known that the group of isomorphisms of every abelian group is the direct product of the groups of isomorphisms of its Sylow subgroups, and hence it results that the group of isomorphisms of any abelian group is completely determined by the groups of isomorphisms of its Sylow subgroups. It will therefore be assumed in what follows that the abelian group \( G \) under consideration has an order which is of the form \( p^m \), \( p \) being a prime number. It will be convenient to represent the characteristic subgroups of \( G \) which are generated by all of its operators of order \( p \), \( p^2 \), \( p^3 \), \ldots, \( p^r \) by \( H_1 \), \( H_2 \), \ldots, \( H_r \), respectively, where \( p^r \) represents the largest order of an operator contained in \( G \). Hence it results that if the order of \( H_a \) divided by the order of \( H_{a-1} \), \( a \leq l \), is \( p^\delta \) then \( G \) has exactly \( \beta \) invariants which exceed \( p^{r-1} \). In particular, if \( H_0 \) is the identity and if the order of \( H_1 \) is \( p^7 \) then the total number of invariants of \( G \) is \( \gamma \).

Since every abelian group is generated by its operators of highest order and all of these operators are conjugate under its group of isomorphisms it results that this group can be represented as a transitive substitution group on letters corresponding to the operators of highest order in the abelian group. We shall first consider the invariant subgroup of this transitive substitution group for \( G \), which corresponds to all the automorphisms of \( G \) in which all the operators of \( H_{l-1} \) are left invariant. All of the transitive constituents of this invariant subgroup are of degree \( p^7 \) since every operator of highest order in \( G \) may correspond to itself multiplied by any operator of \( H_l \) under the automorphisms in question. The degree of every substitution except the identity contained in this subgroup is \( p^m - p^{m-1} \) and the degree of the subgroup is \( p^m - p^{m-\delta} \), where \( \delta \) represents the number of the largest invariants of \( G \). Moreover, this subgroup is the direct product of \( \delta \) substitution groups of order \( p^7 \) and of degree \( p^m - p^{m-1} \), each of which is a simple isomorphism between regular abelian groups of order \( p^7 \) and of type \( (1, 1, 1, \ldots) \). That is, the group of isomorphisms of every abelian group of order \( p^m \) which involves operators of order \( p^8 \) contains an invariant abelian subgroup of order \( p^{8\gamma} \) and of type \( (1, 1, 1, \ldots) \), where \( \delta \) and \( \gamma \) represent the number of largest invariants and the total number of invariants, respectively.

In the study of the group of isomorphisms of \( G \) it is sometimes desirable to consider also the substitution groups on the letters corresponding to the operators of the same order when these operators are not of highest
order. A necessary and sufficient condition that at least one of these substitution groups is transitive is that all the invariants of $G$ are equal to each other and when this condition is satisfied all of these substitution groups are transitive. That is, when $G$ has different invariants, then all the operators of the same order cannot be conjugate under the group of isomorphisms of $G$ unless this order is a maximum for $G$. For our present purpose it is, however, more important to note that the orders of these substitution groups increase with the order of the operators to which they correspond, and that the invariant subgroup of the group corresponding to the operators of order $p^a$ which corresponds to the identity of the group corresponding to the operators of order $p^{a-1}$, $a > 1$, is abelian and of type $(1, 1, 1, \ldots)$. Hence it results that the group of isomorphisms of every abelian group of order $p^m$ which involves operators of order $p^a$ contains an invariant subgroup whose order is of the form $p^b$, composed of all its operators which leave invariant every operator of order $p^a$ contained in this abelian group, where $\alpha$ varies from 1 to $l - 1$.

From this theorem it results directly that the substitution group which corresponds to the permutations of the operators of order $p$ in $H_1$ must involve all of the factors of composition of the group of isomorphisms of $G$ except some of those which are equal to $p$. This substitution group has a number of transitive constituents which is equal to the number of the different invariants of $G$, and the group of isomorphisms of $G$ is solvable whenever this substitution group is solvable, and vice versa. It is therefore easy to determine therefrom a necessary and sufficient condition that the group of isomorphisms of $G$ is solvable, since the group of isomorphisms of the abelian group of order $p^a$ and of type $(1, 1, 1, \ldots)$ is insolvable whenever $\alpha > 1$ and $p > 3$, and also whenever $\alpha > 2$ and $p = 2$ or 3. Hence there results the following theorem in regard to the solvability of the group of isomorphisms of any abelian group of order $p^m$: A necessary and sufficient condition that the group of isomorphisms of an abelian group of order $p^m$, $p$ being a prime number, is solvable is that no two of its invariants are equal to each other when $p > 3$, and that at most two of these invariants are equal to each other whenever $p$ is 2 or 3.

It results directly from the preceding developments that a necessary and sufficient condition that the group of isomorphisms of an abelian group of order $p^m$ contains only one Sylow subgroup whose order is a power of $p$ is that no two of its invariants are equal to each other. When this condition is satisfied the order of the group of isomorphisms of $G$ is the product of the order of this Sylow subgroup and $(p - 1)^\gamma$. In particular, a necessary and sufficient condition that the order of the group of isomorphisms of the abelian group of order $2^m$ is a power of 2 is that no two of the invariants of this abelian group are equal to each other.
When the invariants of such a $G$ vary from $p$ to $p^\gamma$ the order of $H_k$, $k \leq \gamma$, is $p$ with the exponent

$$k\gamma = \frac{k(k - 1)}{2}$$

and the number of the characteristic subgroups whose largest invariant is $p^h$ is the $(\gamma - k + 1)^{th}$ figurate number of order $k$. A necessary and sufficient condition that any abelian group of order $p^m$ contains a cyclic characteristic subgroup of order $p^\alpha$ is that it contains only one largest invariant and that the difference between the indices of $p$ which represent the values of two largest invariants is at least equal to $\alpha$.

When all of the invariants of $G$ are equal to each other then the invariant subgroup of the group of isomorphisms corresponding to the operators of order $p^\alpha$ which corresponds to the identity of the group of isomorphisms corresponding to the operators of order $p^\gamma - 1$, $\alpha > 1$, is of order $p^{\gamma - 1}$, and is the direct product of substitution groups of degree $p^{\gamma - 1}$ and of type $(1, 1, 1, \ldots)$, for every value of $\alpha$ from 2 to $l$. In this special case the order of the invariant subgroup of order $p^\beta$ which is composed of all the automorphisms of $G$ which leave invariant its operators of order $p$ is therefore readily determined by means of the orders of these invariant subgroups, and $\beta = \gamma^2(l - 1)$. Moreover, it follows directly from these correspondencies that the order of the largest operator in this subgroup of order $p^\beta$ cannot exceed $p^{\gamma - 1}$, and this order is obviously actually attained when $p > 2$, and also when $p = 2$ and $\gamma > 1$. These results can readily be applied to the groups of isomorphisms of any abelian group. In particular, a necessary and sufficient condition that the group of isomorphisms of any abelian group is solvable is that it has either at least two invariants which are equal to $p^\alpha$ when $p > 3$, or at least three such invariants when $p$ is either 2 or 3, when the invariants of the abelian group are so chosen that each is a power of some prime number $p$.

In view of the simplicity of the category of groups all of whose operators can be represented in the form $s^t^p$, where $s$ and $t$ are two generating operators of the groups, it seems desirable to note here the exact form of a fundamental theorem relating thereto which was stated inaccurately in my article published in these PROCEEDINGS, volume 13 (1927), page 759, as well as in the abstract of this article which appeared in the Fortschrifte der Mathematik, volume 53 (1930), page 106. This theorem should read as follows: A necessary and sufficient condition that all the operators of a group which is generated by $s$ and $t$ can be represented in the form $s^t^p$ is that the largest subgroup of at least one of the two subgroups generated by $s$ and $t$, respectively, which is invariant under the entire group, gives rise to a quotient group in which the subgroup corresponding to the other is invariant.
It may be desirable to add here that when these largest invariant subgroups of the two groups generated by \( s \) and \( t \), respectively, are of the same index under these cyclic groups then the quotient group with respect to the invariant group generated by these subgroups is the direct product of their complementary cyclic groups. Moreover, these two invariant cyclic subgroups must always generate an abelian group when the cyclic groups generated by \( s \) and \( t \), respectively, have only the identity in common. That is, when the order of this group is a maximum. The fact that the term "un teorema del Chapman," which appears on the page preceding the one of the *Fortschritte der Mathematik* to which we referred in the preceding paragraph, is a misnomer was noted by Chapman himself in the *Messenger of Mathematics*, volume 43 (1914), page 85. This correction seems especially desirable since the theorem is quite elementary.

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**Coalescence of Parts of a Complex**

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Communicated April 18, 1930

1. *Introduction.*—We shall obtain relations involving the changes in Betti numbers resulting when a number of homeomorphic parts of a complex are made to coincide. Related questions have been treated by W. Mayer, L. Vietoris and the writer.

Notations in analysis situs will be as given in a paper by Alexander, with the terms closed chain and connectivity number replaced by cycle and Betti number, respectively. The proofs and results hold for Betti numbers taken either absolute or modulo 2. We use \( R_i(K) \) to denote the \( i \)th Betti number of \( K \).

2. *Corresponding Chains.*—The main theorem follows. Its proof will require a few more sections.

**Theorem 1.** Let \( A \) be a given \( n \)-complex, and \( B_0, B_1, \ldots, B_k \) sub-complexes such that no two have a point in common and each two are homeomorphic under a cell-to-cell correspondence. Let \( D \) be the complex obtained by regarding corresponding points as identical. Then non-negative integers \( \alpha_i, i = 0, 1, \ldots, n; \) and \( \beta_j, j = 0, 1, \ldots, n + 1; \) exist, with \( \beta_0 = \beta_{n+1} = 0, \) such that

\[
kR_i(B_0) = \beta_{i+1} + \alpha_i, \tag{2.1}
\]

\[
R_i(D) - R_i(A) = \beta_i - \alpha_i, \quad i = 0, 1, \ldots, n. \tag{2.2}
\]