1. The quantities \( g_{ij} \) are the nonsymmetric components of the metric tensor of the space of \( n \) dimensions.

The geometry is based upon the equations

\[
g_{ij,k} = g_{a} \Gamma_{jk}^{a} + g_{b} \Gamma_{ik}^{a}.
\]

Equation (1) is a generalization of the equation

\[
g_{ij,k} = g_{a} \{j_{k}^{i} \} + g_{b} \{i_{k}^{j} \}
\]

of Riemannian geometry,\(^1\) in which \( \{j_{k}^{i} \} \) and \( \{i_{k}^{j} \} \) are Christoffel symbols of the second kind in the symmetric \( g_{ij} \).

2. For

\[
g_{ij} = g_{ij} + g_{ji},
\]

where \( g_{ij} \) and \( g_{ji} \) denote the symmetric and skew-symmetric parts of \( g_{ij} \), equation (1) becomes

\[
g_{ij,k} + g_{ij,k} = g_{a} \Gamma_{jk}^{a} + g_{b} \Gamma_{ik}^{a}.
\]

Equation (4) is a generalization of the equation

\[
g_{ij,k} = g_{a} \{j_{k}^{i} \} + g_{b} \{i_{k}^{j} \}
\]

of Riemannian geometry, in which \( \{j_{k}^{i} \} \) and \( \{i_{k}^{j} \} \) are Christoffel symbols of the second kind in the symmetric \( g_{ij} \).

In accordance with equation (3), the equation of minimal curves, that is, curves of length zero, is

\[
g_{ij} \frac{dx^{i}}{ds} \frac{dx^{j}}{ds} = 0,
\]

which is a generalization of the equation

\[
\frac{d^{2}x^{i}}{ds^{2}} + \Gamma^{i}_{jk} \frac{dx^{j}}{ds} \frac{dx^{k}}{ds} = 0,
\]
of regular Riemannian geometry.\(^2\)

When equation (7) is differentiated with respect to \(s\), the resulting equation is with change of certain repeated indices

\[
g_{ij,k} \frac{dx^i}{ds} \frac{dx^j}{ds} + \frac{1}{4} g_{ij} \frac{d^2 x^i}{ds^2} + \frac{1}{4} g_{kj} \frac{d^2 x^k}{ds^2} = 0. \tag{9}
\]

By means of equation (6) and equations of the type (8), this equation becomes

\[
(g_{ik} \Gamma^k_{jk} + g_{jk} \Gamma^k_{ik}) \frac{dx^i}{ds} \frac{dx^j}{ds} - \frac{1}{4} g_{ij} \frac{d^2 x^i}{ds^2} - \frac{1}{4} g_{kj} \frac{d^2 x^k}{ds^2} = 0.
\]

By change of certain repeated upper and lower indices, this equation becomes

\[
(g_{ik} \Gamma^k_{jk} + g_{jk} \Gamma^k_{ik} - g_{ij} \Gamma^i_{jk} - g_{jk} \Gamma^i_{ik}) \frac{dx^i}{ds} \frac{dx^j}{ds} = 0.
\]

Since this equation is satisfied identically, it follows that the geodesics of this generalized geometry are minimal curves.

3. Spaces of positive and negative constant curvature in regular Riemannian geometry admit minimal geodesics with equations of the type (9), but not in other general spaces.

When equation (1) is differentiated with respect to \(x^i\), the resulting equation is

\[
g_{ij,k,l} = g_{ik} \Gamma^k_{jk,l} + g_{jk} \Gamma^k_{ik,l} + \Gamma^k_{j} \Gamma^l_{ik} + \Gamma^k_{i} \Gamma^l_{jk}.
\]

When \(g_{ik,j} \) and \(g_{kj,i} \) are replaced by

\[
g_{ik} \Gamma^m_{hl} + g_{ml} \Gamma^m_{hl},
\]

\[
g_{hl} \Gamma^m_{ji} + g_{mj} \Gamma^m_{hl},
\]
in accordance with equation (1), equation (10) becomes

\[
g_{ij,k,l} = g_{ik} (\Gamma^k_{j} \Gamma^l_{ik} + \Gamma^m_{jl} \Gamma^l_{mi}) - g_{jk} (\Gamma^k_{i} \Gamma^l_{jk} + \Gamma^m_{il} \Gamma^l_{mi}) + g_{ml} \Gamma^m_{jl} \Gamma^l_{mi} + g_{ml} \Gamma^m_{ij} \Gamma^l_{ml}.
\]

When \(k \) and \(l \) are interchanged, the resulting equation is

\[
g_{ij,k,l} = g_{ik} (\Gamma^k_{j} \Gamma^l_{ik} + \Gamma^m_{jl} \Gamma^l_{mi}) + g_{jk} (\Gamma^k_{i} \Gamma^l_{jk} + \Gamma^m_{il} \Gamma^l_{mi}) + g_{ml} \Gamma^m_{jl} \Gamma^l_{mi} + g_{ml} \Gamma^m_{ij} \Gamma^l_{ml}.
\]

When equation (12) is subtracted from equation (11), the resulting equation is

\[
g_{ij} (\Gamma^m_{jl,k} + \Gamma^m_{jk} \Gamma^l_{ml} - \Gamma^m_{jl} \Gamma^l_{mk}) + g_{kl} (\Gamma^m_{ik,l} + \Gamma^m_{li} \Gamma^l_{mk} - \Gamma^m_{il} \Gamma^l_{mk}) = 0.
\]

In terms of quantities of the type

\[
R^i_{jkl} = \Gamma^i_{jl,k} - \Gamma^i_{jk,l} + \Gamma^m_{jl} \Gamma^i_{mk} - \Gamma^m_{jk} \Gamma^i_{ml},
\]

which is a generalization of the equation
\[ R_{ijkl}^h = \frac{\partial}{\partial x^k} \{ h \}_{ij}^l - \frac{\partial}{\partial x^l} \{ h \}_{jk}^i + [m] \{ h \}_{ij}^l - [m] \{ h \}_{jk}^i \]

in regular Riemannian geometry,\(^3\) equation (13) becomes minus one times the equation

\[ g_{ih}R^h_{jk} + g_{jh}R^h_{ik} = 0. \quad (15) \]

Since \( R^h_{ik} \) is skew-symmetric in \( k \) and \( l \), equation (15) is equal to the equation

\[ g_{ih}R^h_{jk} = g_{jh}R^h_{ik}. \quad (16) \]

For \( g^{ih} \), where

\[ g^{ih}g_{ih} = 1 \quad (17) \]

with \( i \) summed from 1 to \( n \) and \( h \) not summed, when equation (16) multiplied by \( g^{ih} \) and summed for \( i \), the resulting equation is

\[ R^h_{jk} = g^{ih}g_{jm}R^m_{ik}. \]

when \( l \) is replaced by \( h \), the resulting equation is

\[ R^h_{jk} = g^{ih}g_{jm}R^m_{ik}, \]

which is

\[ R_{jk} = g^{ih}g_{jm}R^m_{ik}, \quad (18) \]

where \( R_{jk} \) is a generalization in regular Riemannian geometry.\(^4\)

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**THE SOLUTION OF SETS OF EQUATIONS IN GROUPS**

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This paper contains:

1. A generalization of the theorem of Hopf\(^1\) which asserts that the degree of the mapping \( X \rightarrow X^d \) of a compact connected Lie group of rank \( m \) into itself is \( d^m \).

2. A fundamental extension theorem for finite groups having the following as a special case. If \( G \) is a finite group, \( g_1, \ldots, g_n \) arbitrary elements of \( G \), and \( S_1, \ldots, S_n \) any integers such that \( \Sigma S_i \neq 0 \), then there exists a finite extension of \( G \) in which the equation \( x^{S_1}g_1x^{S_2}g_2 \ldots x^{S_n}g_n = 1 \) has a solution, and

3. A partial generalization of the "Freiheitssatz" of group theory.

The proof of the extension theorem rests heavily on Hopf's theorem and basic propositions of algebraic geometry and algebraic number theory. No satisfactory algorithm is known which constructs, in every case, an extension of the type as-