$S - \overline{D}$ is connected and its boundary is $\overline{D}$. But $M \cdot K$ is the uncountable totally disconnected point set $\beta$.

Theorem 17 does not remain true if the requirement that $K$ be compact is omitted. To see this, interchange $M$ and $K$ in the description of Example 2 of C. B. It is clear from Example 3 that it does not remain true on the omission of the requirement that $M$ be compact.

2. These Proceedings, 28, 550–555 (1942). This paper will be referred to as “C. B.”
4. These Proceedings, 28, 555–561 (1942). This paper will be referred to as “D. C. B.” If the proof of Theorem 6 of this paper is modified only to the extent of replacing the definition of $H_n$ by the requirement that (1) for each $n$ greater than 1, $H_n$ be defined as it is there defined for each $n$ and (2) $H_1$ be a collection of connected domains properly covering $K$ such that each of them lies in some region of $G_1$ and every one of them that intersects $M$ intersects $M - K$ and no one of them that intersects $H$ has a point in common with any one of them that intersects $L$, then the arc $C$ necessarily lies, except for its end-points, wholly in some component of $S - M$ that contains points of $K$.

CONCERNING A CONTINUUM AND ITS BOUNDARY

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In this paper some theorems will be established concerning certain relationships between the boundary of a continuum and the components of that continuum minus its boundary. Numbered axioms and chapters herein referred to are axioms and chapters of the author's book "Foundations of Point Set Theory."  

Theorem 1. If, in a space satisfying Axioms 0 and 1, $\beta$ is the boundary of the compact continuum $M$ and, for every component $D$ of $M - \beta$, $T_D$ is a connected point set lying in $M$ and containing the common part of $\beta$ and the boundary of $D$, and $T$ is the sum of all the point sets $T_D$ for all such $D$'s, then $\beta + T$ is connected.

Proof. Suppose $\beta + T$ is the sum of two mutually separated point sets $H$ and $K$. The point sets $H \cdot \beta$ and $K \cdot \beta$ exist and are mutually exclusive and closed. There exists a subcontinuum $N$ of $M$ which is irreducible from $H \cdot \beta$ to $K \cdot \beta$. The connected point set $N - (H \cdot \beta + K \cdot \beta)$ is a subset of some component $L$ of $M - \beta$. Each of the point sets $H \cdot \beta$ and $K \cdot \beta$ con-
tains a limit point of \( N = (H \cdot \beta + K \cdot \beta) \) and therefore of \( L \). Hence \( T_L \)
intersects both \( H \cdot \beta \) and \( K \cdot \beta \). This involves a contradiction.

**THEOREM 2.** Theorem 1 remains true if, in the statement of its hypothesis, the
requirement that \( M \) be compact is replaced by the requirement that space satisfy Axiom 2.

**Proof.** Suppose \( \beta + T \) is the sum of two mutually separated point sets \( H_1 \) and \( H_2 \).
Suppose \( D \) is a component of \( M - \beta \). Since the connected point set \( T_D \) is a subset of \( H_1 + H_2 \), it is a subset either of \( H_1 \) or of \( H_2 \). If there exists at least one component \( D \) of \( M - \beta \) such that \( T_D \) is a subset of \( H_1 \), let \( H_1' \) denote \( H_1 \) plus the sum of all such point sets \( D \). Otherwise let \( H_1' \) denote \( H_1 \). Since Axiom 2 holds true the components of \( M - \beta \) are all domains and, by Theorem 2(\( \xi \)) of Chapter II, every limit point of their sum belongs either to one of them or to the closure of the sum of their boundaries. But, since they are domains, no one of them contains a limit point of its complement, and the boundary of every one of them is a subset of \( \beta \). It follows that if the point \( P \) of \( H_2' \) is a limit point of \( H_1' \), then \( P \) belongs to \( H_2 \) and is a limit point of \( H_1 \). This is impossible. Similarly \( H_1' \) contains no limit point of \( H_2' \). Thus \( H_1' \) and \( H_2' \) are mutually separated. But their sum is the continuum \( M \). This involves a contradiction.

**THEOREM 3.** If, in a space satisfying Axioms 0 and 1, \( \beta \) is the boundary
of the compact continuum \( M \) and, for every component \( D \) of \( M - \beta \), the common part of \( \beta \) and the boundary of \( D \) is connected, then \( \beta \) is connected.

**THEOREM 4.** If, in a space satisfying Axioms 0, 1 and 2, \( \beta \) is the boundary
of the continuum \( M \) and, for every component \( D \) of \( M - \beta \), the common part of \( \beta \) and the boundary of \( D \) is connected, then \( \beta \) is connected.

Theorems 3 and 4 are corollaries of Theorems 1 and 2, respectively.

If Axioms 0, 1 and 2 hold true and \( D \) is a component of the continuum \( M 
\) minus its boundary \( \beta \) then \( D \) is a domain and its boundary is a subset of \( \beta \).
Hence Theorem 4 remains true if, in the statement of its hypothesis, “the common part of \( \beta \) and the boundary of \( D \)” is replaced by “the boundary of \( D \).” But this is not true of Theorem 3 even though the resulting hypothesis is strengthened by the addition of the stipulation that space is compact.
Consider the following example.

**Example 1.** In a Cartesian plane \( E \) let \( Q \) denote a compact totally
disconnected closed point set lying on the \( Y \)-axis, let \( A \) and \( B \) denote the
points \((-1, 0)\) and \((-2, 0)\), respectively, let \( S \) denote the sum of the
straight line interval \( AB \) and all straight line intervals with one end-point at \( A \)
and mid-point at a point of \( Q \), let \( M \) denote the sum of all straight-line
intervals with one end-point at \( A \) and the other one at a point of \( Q \) and let\( \beta \) denote the point set \( A + Q \). Let \( \Sigma \) denote the subspace of \( E \) whose
points are the points of \( S \). In the compact space \( \Sigma \), the boundary of
the continuum \( M \) is \( \beta \) and if \( D \) is a component of \( M - \beta \) then the boundary of
\( D \) is the continuum \( \bar{D} \). But \( \beta \) is not connected.
That Theorem 3 does not remain true if the stipulation that \( M \) is compact is replaced by the stipulation that \( \beta \) is compact may be seen from the following example.

**Example 2.** In a Cartesian plane \( E \), let \( O \) and \( A \) denote the points \((0, 0)\) and \((0, 2)\), respectively, and, for each \( n \), let \( B_n \) denote \((-1/n, 0)\). Let \( M \) denote the sum of \( O \) and the straight-line intervals \( AB_1, AB_2, AB_3, \ldots \) let \( K \) denote a semi-circle with extremities at \( A \) and \( O \) and containing the point \((1, 1)\). Let \( S \) denote \( M + K \). Let \( E \) denote the subspace of \( E \) whose points are the points of \( S \). Here \( \beta \) is \( A + O \) and, for each component \( D \) of \( M - \beta \), \( \beta \cdot D \) is \( A \).

**Theorem 5.** If, in a space satisfying Axioms 0, 1 and 2, \( \beta \), the boundary of the continuum \( M \) is the sum of two mutually exclusive closed point sets \( \beta_1 \) and \( \beta_2 \), then there is a component of \( M - \beta \) whose closure intersects both \( \beta_1 \) and \( \beta_2 \).

**Proof.** The closure of every component of \( M - \beta \) intersects \( \beta \) and therefore either \( \beta_1 \) or \( \beta_2 \). Let \( M_1 \) denote the set of all points that belong either to \( \beta_1 \) or to some component of \( M - \beta \) whose closure intersects \( \beta_1 \). Since their sum is the connected point set \( M_1 \) and \( M_2 \) are not mutually separated. Suppose they are mutually exclusive. Then one of them contains a limit point of the other one. But every component of \( M - \beta \) is a domain and no domain contains a limit point of its complement. Hence either \( \beta_1 \) contains a limit point of \( M_2 \) or \( \beta_2 \) contains a limit point of \( M_1 \). Suppose \( \beta_1 \) contains a point \( P \) which is a limit point of \( M_2 \). Let \( Q \) denote the collection of all components of \( M - \beta \) whose closures intersect \( \beta_2 \). Since it is not a limit point of \( \beta_2 \), \( P \) must be a limit point of \( Q^* \). Hence, by Theorem 2(b) of Chapter 11, \( P \) belongs to the closure of the sum of the boundaries of the domains of the collection \( Q \). Since the boundary of every domain of \( Q \) belongs to the closed point set \( \beta_2 \) it follows that \( P \) belongs to \( \beta_2 \) and therefore that \( \beta_1 \) intersects \( \beta_2 \). The same result is obtained in case \( \beta_2 \) contains a limit point of \( \beta_1 \). Thus the supposition that \( M_1 \) and \( M_2 \) are mutually exclusive leads to a contradiction. It follows that there is a component of \( M - \beta \) whose closure intersects \( \beta_1 \) and \( \beta_2 \).

**Definition.** If \( H, L \) and \( N \) are subsets of the continuum \( M \), \( N \) is said to shield \( H \) from \( L \) in \( M \) if \( N \) has no point in common with \( L \) but intersects every connected subset of \( M \) which intersects both \( H \) and \( L \).

**Theorem 6.** If \( A \) and \( B \) are points of the continuum \( M \), \( T \) is a compact subcontinuum of \( M \) containing \( A \) and \( B \), \( H \) is a closed and compact subset of \( M \) and, for each point \( P \) of \( T - (A + B + T \cdot H) \), there is a continuum lying in \( H \) and shielding \( A + B \) from \( P \) in \( M \) then \( A \) and \( B \) belong to the same component of \( H \).

**Proof.** Suppose \( A \) and \( B \) do not lie in the same component of \( H \). There exists a well-ordered sequence \( \alpha \) whose terms are the points of \( T - (A + B + T \cdot H) \), a well-ordered sequence \( \beta_1 \) whose terms are the continua which...
are subsets of $H$, and a well-ordered sequence $\beta_2$ whose terms are the continua that lie in $T + H$. Suppose $T$ is not a subset of $H$. There exists a well-ordered sequence $\gamma$ such that (1) $T$ is the first term of $\gamma$, (2) each term of $\gamma$ is a continuum lying in $T + H$ but not wholly in $H$ and (3) if $x$ is a term of $\gamma$ and $P_x$ is the first point of $\alpha$ belonging to $x$ and $H_x$ is the first term of $\beta_1$ which shields $A + B$ from $P_x$ in $M$, $T_{A_x}$ is the first term of $\beta_2$ which is an irreducible continuum from $A$ to $H_x$ lying in $x$ and $T_{B_x}$ is the first term of $\beta_2$ which is an irreducible continuum from $B$ to $H_x$ lying in $x$, then $x$ is the last term of $\gamma$ or $H_x + T_{A_x} + T_{B_x}$ is the first term following $x$ in $\gamma$, according as $T_{A_x} + T_{B_x}$ is, or is not, a subset of $H$, (4) if $\gamma'$ is a well-ordered proper subsequence of $\gamma$ with no last term and $\gamma'$ has, as one of its terms, each term of $\gamma$ which, in $\gamma$, precedes a term of $\gamma'$, then the limiting set of the sequence $\gamma'$ is the first term of $\gamma$ which, in $\gamma$, follows all the terms of $\gamma'$, unless this limiting set is a subset of $H$ in which case $\gamma'$ is $\gamma$.

Let $\gamma_T$ denote a well-ordered sequence such that (1) $x$ is a term of $\gamma_T$ if, and only if, $x$ is the common part of $T - T^\cdot H$ and some continuum which is a term of $\gamma$, (2) $x$ precedes $y$ in $\gamma_T$ if, and only if, there exist terms $x'$ and $y'$ of $\gamma$ such that $x = x^\cdot T$ and $y = y^\cdot T$ and $x'$ precedes $y'$ in $\gamma$. Let $\gamma_H$ denote a well-ordered sequence described in exactly the same manner except for the substitution of $\gamma_H$ for $\gamma_T$ and of $H$ for $T - T^\cdot H$. Let $Q, Q_T$ and $Q_H$ denote the limiting sets of $\gamma, \gamma_T$ and $\gamma_H$, respectively. It may be seen that $Q = Q_T + Q_H$. Suppose $Q_T$ is not a subset of $H$. Every term of $\gamma_T$ contains every one that follows it and therefore $Q_T - H \cdot Q_T$ is the common part of all the point sets of this sequence. Let $P$ denote the first point of $Q_T$ in the sequence $\alpha$. Let $\omega$ denote a subsequence of $\gamma$ such that $x$ belongs to $\omega$ if, and only if, it contains a point that precedes $P$ in $\alpha$. The second term of $\gamma$ that follows every term of $\omega$ exists but does not contain $P$. This involves a contradiction.

Theorem 6 does not remain true if the stipulation that, for each point $P$ of $T - (A + B + T^\cdot H)$ there is a continuum lying in $H$ and shielding $A + B$ from $P$ is replaced by the stipulation that for each such point $P$ there is a continuum lying in $H$ and intersecting every subcontinuum of $M$ which intersects both $P$ and $A + B$. Consider the following example.

**Example 3.** In a Euclidean plane, let $M$ denote a compact indecomposable continuum and let $A$ and $B$ denote points belonging to different components of $M$. Let $H$ denote $A + B$. If $P$ is a point of $M - H$, $M$ is an irreducible continuum either from $P$ to $A$ or from $P$ to $B$. In the first case, $B$ intersects every subcontinuum of $M$ that intersects both $P$ and $A + B$. In the second case, $A$ intersects every such continuum. But no component of $H$ contains both $A$ and $B$.

The following theorem may be easily proved with the assistance of Theorem 6.
Theorem 7. If the points $A$ and $B$ belong to different components of $\beta$, the boundary of the compact continuum $M$, there exists a component $D$ of $M - \beta$ such that no component of $\beta$ shields $A + B$ from $D$ in $M$.

Theorem 8. If, in a space satisfying Axioms 0 and 1, $\beta$ is the boundary of the continuum $M$, $D$ is a component of $M - \beta$ and $K$ is a continuum intersecting $D$ but not lying wholly in it and $M \cdot K$ is compact then every component of $K \cdot D$ contains a point of $\beta$.

Proof. Suppose there exists a component $N$ of $K \cdot D$ containing no point of $\beta$. Then there exists a subset $H$ of $K \cdot D \cdot (S - \beta)$ containing $N$ such that either $N = H$ or $H$ and $K \cdot D - H$ are mutually exclusive and closed. There exists a domain $I$ containing $H$ such that $I \cdot K \cdot D = H$ and $I \cdot (\beta + S - M)$ is vacuous. There exists a subcontinuum $T$ of $K$ lying in $I$ and intersecting both $N$ and $S - I$. The point set $T$ is a connected subset of $K$ intersecting $N$ but not lying wholly in it. Hence it is not a subset of $D$. But it is a subset of $M - \beta$ and it contains a point of $D$. This involves a contradiction.

Theorem 8 does not remain true if the stipulation that $M \cdot K$ is compact is replaced by the stipulation that $D$ is compact and that $M$ is identical with $K$. Consider the following example.

Example 4. In a Cartesian plane $E$, let $A$ and $D$ denote the points $(0, 2)$ and $(0, 1)$ and, for each $n$, let $B_n$ denote the point $(-1/n, 0)$ and let $AB_n$ denote the straight-line interval whose end-points are $A$ and $B_n$. Let both $K$ and $M$ denote the point set $D + AB_1 + AB_3 + \ldots$ and let $S$ denote $M + AB_1$. Let $\Sigma$ denote the subspace of $E$ whose points are the points of $S$. Here $\beta$ is $A$ and the point $D$ is a component of $M - \beta$. Though $D$ is compact, the component $D$ of $K \cdot D$ does not contain $A$.

That Theorem 8 does not remain true if the requirement that $M \cdot K$ be compact is replaced by the requirement that space be the plane and $D$ be the only component of $M - \beta$ may be seen from the following example.

Example 5. In a Cartesian plane $E$, let $M$ denote the set of all points $(x, y)$ such that either $-1/x^2 \leq y \leq 1/x^2$ or $x = 0$ and let $K$ denote the set of all points $(x, y)$ such that either $x = 0$ or $y = (2 \sin 1/x^2)/x^2$.

However, Theorem 8 does remain true if the requirement that $M \cdot K$ be compact is replaced by the requirement that Axiom 2 hold true and $K \cdot D$ be compact.

Theorem 9. If, in a space satisfying Axioms 0 and 1, $\beta$ is the boundary of the continuum $M$, $D$ is a component of $M - \beta$ and $K$ is a continuum intersecting $D$ and such that $M \cdot K$ is compact and either $\beta \cdot D$ is vacuous or there exists a subcontinuum of $K$ lying in $D$ and containing $\beta \cdot D$, then $K \cdot D$ is a continuum.

Theorem 9 may be easily established with the help of Theorem 8. It does not remain true if $M \cdot K$ is replaced by $D$. Consider the following example.
Example 6. Let space be that of Example 2, but now let \( K \) denote the point set \( AB_2 + AB_3 + \ldots \), let \( M \) denote \( A + (S - AB_2) \) and let \( D \) denote \( S - (AB_1 + AB_2 + \ldots) \). The point set \( M \cdot K \) is now \( K, \overrightarrow{D} \) is compact and \( \beta \), the boundary of \( M \), is the point \( A \) which is a subcontinuum of \( K \) lying in \( \overrightarrow{D} \) and coinciding with \( \beta \cdot \overrightarrow{D} \). But \( K \cdot \overrightarrow{D} \) is \( A + O \) which is not connected.

Theorem 9 does not remain true if the stipulation that \( M \cdot K \) is compact is replaced by the stipulation that \( \overrightarrow{D} \) is compact even though the resulting hypothesis is strengthened by the elimination of the word "either" and "or there exists a subcontinuum of \( K \) lying in \( \overrightarrow{D} \) and containing \( \beta \cdot \overrightarrow{D} \)." Consider the following example.

Example 7. In a Cartesian plane \( E \), let \( A, A_1 \) and \( A_2 \) denote the points \((0, 4), (0, 1)\) and \((0, 3)\), respectively and, for each \( n \), let \( B_n \) denote \((-1/n, 0)\) and let \( AB_n \) denote the straight-line interval whose extremities are \( A \) and \( B_n \). Let \( T \) denote a semi-circle passing through \((1, 2)\) and with its extremities at \( A_1 \) and \( A_2 \). Let \( K \) denote the point set \( A_1 + A_2 + AB_2 + AB_3 + \ldots \), let \( M \) denote \( K + T \) and let \( S \) denote \( M + AB_1 \). Let \( E \) denote the subspace of \( E \) whose points are the points of \( S \). Here \( D \) is \( T - (A_1 + A_2) \), \( \beta \) is \( A \) and \( \beta \cdot \overrightarrow{D} \) is vacuous. But \( K \cdot \overrightarrow{D} \) is \( A_1 + A_2 \) which is not connected.

Theorem 10. Theorem 9 remains true if the stipulation that \( M \cdot K \) is compact is replaced by the stipulation that Axiom 2 holds true.

Proof. Since Axiom 2 holds true, \( D \) is a domain whose boundary is \( \beta \cdot \overrightarrow{D} \). Hence, by hypothesis, there exists a subcontinuum \( H \) of \( K \) lying in \( \overrightarrow{D} \) and containing \( \beta \cdot \overrightarrow{D} \). If \( K \cdot \overrightarrow{D} \) is neither \( H \) nor \( K \) then \( K - H \) is the sum of the point sets \((K - H) \cdot D \) and \( K - K \cdot D \). Since \( D \) is a domain these point sets are mutually separated. Hence \((K - H) \cdot D + H \) is a continuum. But this point set is \( K \cdot \overrightarrow{D} \).


CONCERNING DOMAINS WHOSE BOUNDARIES ARE COMPACT

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The following theorem may be established by an argument differing slightly from that given to prove Theorem 12 of Chapter IV of the author's "Foundations of Point Set Theory." 1

Theorem 1. If, in a space satisfying Axioms 0–5, \( H \) and \( K \) are two mutually exclusive closed and compact point sets and \( L \) is a closed point set