ON THE CHANGES OF SIGN OF THE DERIVATIVES OF A FUNCTION DEFINED BY A LAPLACE INTEGRAL

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Let a real function \( f(x) \) be defined by a Laplace integral

\[
f(x) = \int_0^\infty e^{-xt} \varphi(t) \, dt,
\]

where \( \varphi(t) \) is a real continuous function in the interval \( 0 \leq t < \infty \), the integral converging for \( x > 0 \). The purpose of this note is to announce certain results concerning the changes of sign of \( f(x) \) and its derivatives as affected by the changes of sign of \( \varphi(t) \).

It was known to Laguerre\(^1\) that \( f(x) \) cannot have more changes of sign than \( \varphi(t) \). Since

\[
f^{(k)}(x) = (-1)^k \int_0^\infty e^{-xt} t^k \varphi(t) \, dt,
\]

it follows that \( f^{(k)}(x) \) cannot have more changes of sign than \( \varphi(t) \). We are able to show that it has exactly as many as \( \varphi(t) \) for all \( k \) sufficiently large. Moreover, we show that if a change of sign of \( \varphi(t) \) is at \( t = a \), then one of the changes of sign of \( f^{(k)}(x) \) will be at a point \( x_k \) such that

\(^1\) E. B. Christoffel, *J. reine angew. Math.* (Crelle), 70, 46 and 241 (1869).


\(^3\) O. Veblen completed the covariant derivative by imposing a set of invariant conditions involving components of conformal tensors. These conditions correspond to our equations (2.13) but differ from these latter equations in the important respect that they involve only derivatives of the first order. See "Differential Invariants and Geometry," *Atti del congresso internazionale dei matematici*, 1, [6] 181–189 (1928); also, "Conformal Tensors and Connections," These PROCEEDINGS, 14, 735–745 (1928).

\(^4\) See, for example, L. P. Eisenhart, *Non-Riemannian Geometry* (New York) 1927, pp. 14–18, where references to the literature are to be found.
In particular, if \( \varphi(t) \) has infinitely many changes of sign, the number experienced by \( f^{(k)}(x) \) will become infinite as \( k \) becomes infinite. We state the theorems leading up to these results. The proofs of the theorems will be published elsewhere.

**THEOREM 1.** The function

\[
(1 + x)^{-k} - e^{-kx}
\]

tends uniformly to zero in the interval \( 0 \leq x < \infty \) as \( k \) becomes infinite.

**COROLLARY.** The function

\[
\left[ \left( \frac{x}{x + n} \right)^{k+1} - e^{-knx} \right] \quad (n = 0, 1, 2, \ldots)
\]

tends uniformly to zero in the interval \( 0 \leq x < \infty \) as \( k \) becomes infinite.

**THEOREM 2.** If \( G_N(x) \) is an exponential polynomial,

\[
G_N(x) = \sum_{n=0}^{\infty} a_n e^{-nt},
\]

and

\[
f(x) = \int_0^\infty e^{-xt} G_N(t) \, dt,
\]

then

\[
\lim_{k \to \infty} \left[ \frac{x^{k+1}}{k!} \left( -1 \right)^k f^{(k)}(x) - G_N \left( \frac{k}{x} \right) \right]
\]

uniformly in the interval \( 0 \leq x < \infty \).

In this theorem and in the preceding corollary the function in brackets is to be defined as zero when \( x = 0 \).

**THEOREM 3.** If \( \varphi(t) \) is continuous in the interval \( 0 \leq t < \infty \), tending to a finite limit as \( t \) becomes infinite, then to an arbitrary positive \( \epsilon \) there corresponds an exponential polynomial

\[
G_N(t) = \sum_{n=0}^{\infty} a_n e^{-nt}
\]

such that

\[
| \varphi(t) - G_N(t) | < \epsilon \quad (0 \leq t < \infty).
\]

**THEOREM 4.** If \( \varphi(t) \) satisfies the conditions of Theorem 3, and if

\[
f(x) = \int_0^\infty e^{-xt} \varphi(t) \, dt,
\]
then
\[ \lim_{k \to \infty} \left[ \frac{x^k + 1}{k!} (-1)^k f^{(k)}(x) - \varphi \left( \frac{k}{x} \right) \right] = 0 \]

uniformly in the interval \( 0 \leq x < \infty \).

**Theorem 5.** If \( \varphi(t) \) satisfies the conditions of Theorem 3, if \( \varphi(t) \) has \( n \) changes of sign in the interval \( 0 < t < \infty \), and if
\[
 f(x) = \int_{0}^{\infty} e^{-xt} \varphi(t) \, dt,
\]
then \( f^{(k)}(x) \) has exactly \( n \) changes of sign in \( 0 < x < \infty \) for \( k \) sufficiently large.

**Corollary.** If \( \varphi(t) \) has infinitely many changes of sign, then \( f^{(k)}(x) \) has \( n_k \) changes of sign in \( 0 < x < \infty \), where
\[
 \lim_{k \to \infty} n_k = \infty.
\]

**Theorem 6.** If \( \varphi(t) \) satisfies the conditions of Theorem 3, has changes of sign at the points \( 0 < t_1 < t_2 < \ldots < t_n \) and at no others, is not identically zero in a neighborhood of any of these points \( t_i \), and if
\[
 f(x) = \int_{0}^{\infty} e^{-xt} \varphi(t) \, dt,
\]
then \( f^{(k)}(x) \) has exactly \( n \) changes of sign for \( k \) sufficiently large at the points
\[
x_{1k} > x_{2k} > \ldots > x_{nk},
\]
and
\[
 \lim_{k \to \infty} \frac{x_{ik}}{k} = \frac{1}{t_i} \quad (i = 1, 2, \ldots, n).
\]

We note that \( \varphi(t) \) satisfies the conditions of this theorem if, in particular, it is analytic in the interval \( 0 \leq t < \infty \) and approaches a finite limit as \( t \) becomes infinite.

The results of Theorems 5 and 6 appear to hold if the restriction that \( \varphi(t) \) tends to a finite limit when \( t \) becomes infinite is omitted. To establish the theorems in this case would involve a change of the present method since the condition cannot be omitted in Theorem 3.

\[ ^1 \text{Laguerre, Oeuvres, 1, p. 28. See also G. Pólya, "Sur un théorème de Laguerre," Compt. Rend. Séances l'Académie Sci., 156, 996 (1913).} \]